ON THE SYMPLECTIC INVARIANCE OF LOG KODAIRA DIMENSION

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ABSTRACT. Suppose that A and B are symplectomorphic smooth affine varieties. If A is acylic of dimension 2 then B has the same log Kodaira dimension as A. If the dimension of A is 3, has log Kodaira dimension 2 and satisfies some other conditions then B cannot be of log general type. We also show that if A and B are symplectomorphic affine varieties of any dimension then any compactification of A by a projective variety is uniruled if and only if any such compactification of B is uniruled.

CONTENTS

1. INTRODUCTION

Much of algebraic geometry is governed by the numerical properties of the canonical class. Other useful properties such as uniruledness and rational connectivity have also played a major role. Also closed smooth projective varieties, and more generally Kähler manifolds have been studied extensively from both a topological and symplectic viewpoint. In complex dimension 2, tools such as Donaldson theory and Seiberg Witten invariants have been used to study such manifolds. For instance in [Wit94] algebraic surfaces of general type have plus or minus their canonical class as diffeomorphism invariants. Plurigenera and hence Kodaira dimension for algebraic surfaces are also shown in [FM97] to be diffeomorphism invariants using Seiberg Witten theory. There are many other results for these surfaces.

One can also study these varieties from a symplectic perspective. We can use tools such as Gromov-Witten theory to see what the symplectic structure tells us about our algebraic variety. Work by Kollár and Ruan ([Kol96] and [Rua99]) tells us the property of being uniruled is a symplectic invariant using Gromov Witten theory. Another extremely useful notion in algebraic geometry is rational connectedness and this has been studied from a symplectic viewpoint in [Voi08] and [Tia12].

Less has been done to study open algebraic varieties from a symplectic perspective, although there has been some work [LZ11]. Also there isn't as much work in higher dimensions although there is some progress in dimension 3 (see for instance [Rua94]). This paper addresses some of these issues. We will be primarily concerned with smooth affine varieties and we will study them from a symplectic perspective. Every smooth affine variety has a symplectic structure coming from some embedding in $\mathbb{C}^N$ and this is a biholomorphic invariant (see [EG91]). A particular algebraic invariant of A is called the log Kodaira dimension. One can ask to what extent is the log Kodaira dimension a symplectic invariant? Log Kodaira dimension is a number $\overline{\kappa}(A)$ which takes values in $\{-\infty, 0, 1, \cdots, \dim_{\mathbb{C}} A\}$. We say that A is of log general type if $\overline{\kappa}(A) = \dim_{\mathbb{C}} A$. A precise definition is given at the start of section 6.

We show that log Kodaira dimension is a symplectic invariant for smooth acyclic affine surfaces (Theorem 6.3). We also show that if A and B are symplectomorphic smooth affine varieties such that:

(1) A has complex dimension 3.
(2) A can be compactified by a smooth normal crossing nef divisor which is linearly equivalent to some smooth divisor
(3) The log Kodaira dimension of A is 2.

then the log Kodaira dimension of B is ≤ 2 (see Theorem 6.11).

A projective variety is *uniruled* if there is a rational curve passing through every point. Let P and Q be smooth projective varieties compactifying smooth affine varieties A and B respectively. We show that if A is symplectomorphic to B and P is uniruled then Q is also uniruled (Theorem 7.1).

In order to prove these theorems we introduce three notions of uniruledness for smooth affine varieties. The first notion is defined for an object called the Liouville domain associated to our affine variety. This is a symplectic invariant and is defined in Section 2. The second notion says that a smooth affine variety is algebraically k uniruled if there is a morphism from $\mathbb{P}^1$ minus at most k points to our variety passing through a generic point and is defined in Section 3. The third notion defined in Section 5 is more flexible than the second notion as it now involves J holomorphic curves from $\mathbb{P}^1$ minus some points where J is any appropriate almost complex structure. This notion is called compactified k uniruled. We show by using degeneration to the normal cone techniques that the first definition implies the second

definition (Theorem 3.3). Also using other simpler techniques we can show that third definition implies the first one (Theorem 5.3). Putting all of this together one gets that if A is symplectomorphic to B and is compactified k uniruled then B is algebraically k uniruled.

Because there is a relationship between log Kodaira dimension and uniruledness in low dimensions (see [MS80],[Miy01], [Kaw79] and [Kis06]) we obtain our log Kodaira dimension results. Similarly if P and Q are projective with symplectomorphic affine open subsets A and B such that P is uniruled, then one can show that A is compactified k uniruled for some k. Hence B is algebraically k uniruled which in turn implies that Q is uniruled.

The paper is organized as follows: In Section 2 we introduce the reader to uniruled Liouville domains (first definition). These are purely symplectic objects. In Section 3 we give a purely algebraic definition of uniruledness for smooth affine varieties (second definition) and relate it to uniruled Liouville domains. In Section 4 we give an introduction to Gromov Witten invariants, then in Section 5 we give a much more flexible definition of uniruledness (third definition) for smooth affine varieties. In Section 6 we use all of the above machinery to prove our log Kodaira dimension invariance results and finally in Section 7 we prove that projective varieties with symplectomorphic open affine subsets are either both uniruled or both not uniruled.

2. Uniruled Liouville domains

Throughout this paper we will use the following notation. If U is any subset of a topological space then we write U^o for the interior of U. Also if (N, ω) is a symplectic manifold and θ is a 1-form then we write X_θ to be the unique vector satisfying $\iota_{X_\theta} \omega = \theta$.

Let M be a compact manifold with boundary with a 1-form θ_M satisfying:

(1) $\omega_M := d\theta_M$ is a symplectic form.
(2) The ω_M-dual X_{θ_M} of θ_M points outwards along ∂M.

We say that (M, θ_M) is a *Liouville domain* if it satisfies the above properties. Let J be an almost complex structure compatible with the symplectic form ω_M. We say that J is a *convex almost complex structure* on M if there is some function $\phi : M \to \mathbb{R}$ so that:
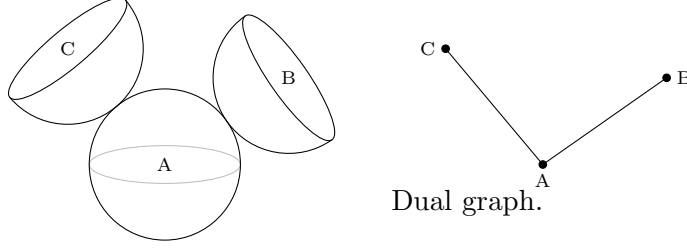
(1) ∂M a regular level set of ϕ and ϕ attains its maximum on ∂M.
(2) $\theta_M \circ J = d\phi$ near ∂M.

Suppose that (N, ω_N) is a symplectic manifold and let J_N be an almost complex structure. If $u : S \to N$ is a J_N-holomorphic map from a Riemann surface S to N then the *energy of u* is defined to be $\int_S u^* \omega_N$.

Definition 2.1. *Let $k > 0$ be an integer and $\lambda > 0$ a real number. We say that a Liouville domain M is (k, Λ)-**uniruled** if for every convex almost complex structure J on M and every point $p \in M^o$ where J is integrable on a neighbourhood of p, there is a proper J holomorphic map $u : S \to M^o$ to the interior M^o of M passing through this point. We require that S is a genus zero Riemann surface, the rank of $H_1(S, \mathbb{Q})$ is at most $k - 1$ and the energy of u is at most Λ.*

Theorem 2.2. *Suppose that N, M are Liouville domains such that M is a codimension 0 symplectic submanifold of N with the property that there exists some 1-form θ' on N so that $\theta'|_M - \theta_M$ is exact and so that $d\theta' = d\theta_N$. If N is (k, Λ)-uniruled then M is also (k, Λ)-uniruled. In particular, the above fact is true if M is a codimension 0 exact submanifold of N or if the inclusion map $M \hookrightarrow N$ is a symplectic embedding and a homotopy equivalence.*

Before we prove this theorem we need some preliminary lemmas and definitions. The following definitions are technically not relevant for the theorem above, but one of the lemmas used in proving this theorem will also be used later on in a slightly more general context. A *nodal Riemann surface* is a 1 dimensional complex analytic variety with the property that the only singularities are nodal. We say that it has *arithmetic genus* 0 if it can be holomorphically embedded into a simply connected compact nodal Riemann surface. An example of an arithmetic genus zero surface is: $B(1) \cap \{z_1 z_2 = 0\} \subset \mathbb{C}^2$ where $B(1)$ is the open unit ball and z_1, z_2 are coordinates for $\mathbb{C}^2$. Note that a genus zero nodal Riemann surface is a union $S_1, \cdots, S_k$ of smooth Riemann surfaces which only intersect each other at the nodal singularities of S. Here $S_1, \cdots, S_k$ are called the *irreducible components of S*. An *arithmetic genus 0 nodal Riemann surface with boundary* is a closed subset S of a compact arithmetic genus 0 nodal Riemann surface C with the property that away from the nodes of C, S is a Riemann surface with boundary. We require that the closure of this boundary does not intersect the nodes of C. This means that the boundary is a union of circles. An example of such a holomorphic object would be the closure of $B(1) \cap \{z_1 z_2 = 0\}$. Again an arithmetic genus 0 nodal Riemann surface with boundary is a union of smooth Riemann surfaces with boundary intersecting each other at the nodal singularities away from their boundaries. These smooth Riemann surfaces with boundary are called the *irreducible components of S*. We can form a graph Γ_S whose nodes are the irreducible components of S and if two irreducible components intersect at some point p then we have an edge E_p joining the appropriate nodes. This is called the *dual graph* of S. The dual graph of every connected arithmetic genus zero compact Riemann surface is a tree.

Riemann surface.   Dual graph.

Lemma 2.3. *Let M be a Liouville domain and N any exact symplectic manifold so that M is a codimension 0 symplectic submanifold of N with the additional property that there is some 1-form θ' on N so that $\theta'|_M - \theta_M$ is exact and so that $d\theta' = d\theta_N$. Let J be a compatible almost complex structure on N so that J restricted to M is a convex almost complex structure. If $u : S \to N^o$ is a J-holomorphic curve with the property that $u^{-1}(M)$ is compact and S is an arithmetic genus zero nodal Riemann surface then the map $H_1(u^{-1}(M^o)) \to H_1(S)$ is injective.*

Proof. of Lemma 2.3. By definition we can choose a collar neighbourhood $(1-\epsilon, 1] \times \partial M$ of ∂M inside M so that $d\theta_M \circ J = dr$ where r parameterizes the interval $(1-\epsilon, 1]$. For $R \in (1-\epsilon, 1)$ we define M_R to be $M \setminus \{r > R\}$. We will show that $H_1(u^{-1}(M_R)) \to H_1(S)$ is injective for generic R and this will prove the theorem because $H_1(u^{-1}(M^o))$ is the direct limit of $H_1(u^{-1}(M_R))$ as R tends to 1.

For generic R, ∂M_R is transverse to u. This means that $S_R := u^{-1}(M_R)$ is an arithmetic genus 0 compact nodal Riemann surface with boundary. Also the closure of $S \setminus S_R$ is a possibly non-compact nodal Riemann surface with boundary equal to ∂S_R. We will write this closure as S_R^c. We let θ be a 1-form on N so that $\theta' - \theta$ is exact and so that $\theta = \theta_M$ on a neighbourhood of M_R. The maximum principle [AS10, Lemma 7.2] using the 1-form θ tells us that every irreducible component of S_R^c is non-compact. Let $S_1', \cdots, S_{l'}'$ be these irreducible components. These are non-compact Riemann surfaces with compact boundary. Hence they have the property that $H_1(\partial S_i') \to H_1(S_i')$ is injective. This in turn implies that $H_1(\partial S_R^c) \to H_1(S_R^c)$ is injective. Because S is the union of S_R and S_R^c along ∂S_R^c, we have by a Mayor-Vietoris argument that the map $H_1(S_R) \to H_1(S)$ is injective. Hence $H_1(u^{-1}(M^o)) \to H_1(S)$ is injective. $\square$

Lemma 2.4. *Every Liouville domain (M, θ_M) has a convex almost complex structure.*

Proof. of Lemma 2.4. By flowing ∂M backwards along X_{θ_M} we have a neighbourhood $(1-\epsilon, 1] \times \partial M$ of M so that $\theta_M = r\alpha_M$ where r parameterizes $(1 - \epsilon, 1]$ and α_M is the contact form $\theta_M|_{\partial M}$. We define the vector bundle V_1 to be the span of the vectors $\frac{\partial}{\partial r}$ and X_{θ_M} and V_2 to be the set of vectors

in the kernal of dr and θ_M. Near ∂M, we have that $TM = V_1 \oplus V_2$ and V_1, V_2 are symplectically orthogonal. We define J so that:

(1) J is compatible with the symplectic form ω_M.
(2) $J(V_1) = V_1$ and $J(V_2) = V_2$ near ∂M. This can be done because these vector spaces are ω_M orthogonal.
(3) $J(r\frac{\partial}{\partial r}) = X_{\theta_M}$.

Here $\theta_M \circ J = dr$ near ∂M and so J is a convex almost complex structure. $\square$

Proof. of Theorem 2.2. Let J be a convex almost complex structure on M. By Lemma 2.4, we have that N admits some convex almost complex structure J_N. Because the space of all almost complex structures compatible with a symplectic form is contractible we can choose a compatible almost complex structure J' so that $J' = J_N$ near ∂N and $J'|_M = J$. Let $p \in M$ be a point in the interior of M such that J is integrable on a neighbourhood of p. Because N is (k, Λ)-uniruled, we have that there exists a proper holomorphic map $u : S \to N^o$ passing through p of energy at most Λ. Also the rank of $H_1(S, \mathbb{Q})$ is at most $k - 1$. By Lemma 2.3, the rank of $H_1(u^{-1}(N^o), \mathbb{Q})$ is also at most $k - 1$. Hence $u|_{u^{-1}(N^o)}$ is a proper J holomorphic map in M^o of energy at most Λ and passing through p, where $|H_1(u^{-1}(N^o), \mathbb{Q})| \leq k - 1$. This implies that M is (k, Λ)-uniruled. $\square$

If (M, θ_M) is a Liouville domain, then by flowing ∂M backwards along X_{θ_M} we have a neighbourhood $(1 - \epsilon, 1] \times \partial M$ of M so that $\theta_M = r\alpha_M$ where r parameterizes $(1 - \epsilon, 1]$ and where α_M is a contact form on ∂M. If we glue $[1, \infty) \times \partial M$ to M along ∂M and extend θ_M by $r\alpha_M$, then we get a new exact symplectic manifold $\widehat{M}$ called the *completion of M*.

Theorem 2.5. *Let M, N be two Liouville domains such that $\widehat{M}$ is symplectomorphic to $\widehat{N}$. If M is (k, Λ)-uniruled then there exists a $\Lambda' > 0$ such that $\widehat{N}$ is (k, Λ')-uniruled.*

Proof. of Theorem 2.5. Let $\phi : \widehat{M} \to \widehat{N}$ by our symplectomorphism. By [BEE12, Lemma 1] we can assume that ϕ is an exact symplectomorphism which means that $\phi^* \theta_N = \theta_M + df$ for some function f. Let $\Phi_t : \widehat{M} \to \widehat{M}$ be the time t flow of the vector field X_{θ_M}. Because X_{θ_M} is equal to $r\frac{\partial}{\partial r}$ near infinity where r is the cylindrical coordinate on $\widehat{M}$, we get that for some $T \geq 0$, $\phi^{-1}(N) \subset \Phi_T(M)$. Because $\Phi_T^* \theta_M = e^T \theta_M$, we get by a rescaling argument that the Liouville domain $\phi_T(M)$ is $(k, e^T \Lambda)$-uniruled. Because N is a codimension 0 exact symplectic submanifold of $\phi_T(M)$, we have by Lemma 2.2 that N is (k, Λ')-uniruled where $\Lambda' = e^T \Lambda$. $\square$

If we have two Liouville domains (M, θ_M) and (N, θ_N) then they are *Liouville deformation equivalent* if there is a diffeomorphism $\phi : M \to N$ and a smooth family of 1-forms θ_M^t ($t \in [0, 1]$) on M with the property that:

(1) $\theta_M^0 = \theta_M$,
(2) $\theta_M^1 = \phi^* \theta_N$, and
(3) (M, θ_M^t) is a Liouville domain for each t.

Corollary 2.6. *Let M, N be two Liouville deformation equivalent Liouville domains. If M is (k, Λ)-uniruled then there exists a $\Lambda' > 0$ such that $\widehat{N}$ is (k, Λ')-uniruled.*

Proof. of Corollary 2.6. We will first show that $\widehat{M}$ is symplectomorphic to $\widehat{N}$ (which is a standard fact) and then we will use Theorem 2.5.

Let θ_M^t be our Liouville deformation on M. By construction, we can complete all the Liouville domains (M, θ_M^t) giving us a manifold $\widehat{M}$ and a smooth family of 1-forms θ_M^t on $\widehat{M}$ by abuse of notation. The vector field X_t given by the $d\theta_M^t$-dual of $\frac{d}{dt}(\theta_M^t)$ is integrable. This is because $dr(X_t)$ is less than or equal to some constant times r where r is the cylindrical coordinate on $\widehat{M}$. The time 1 flow of this vector field gives us a symplectomorphism from $(\widehat{M}, \theta_M^0)$ to $(\widehat{M}, \theta_M^1)$. So because $\theta_M = \theta_M^0$ and $\phi^* \theta_N = \theta_M^1$ we get that $(\widehat{M}, \theta_M)$ is symplectomorphic to $(\widehat{N}, \theta_N)$ Hence by Theorem 2.5, we have that N is (k, Λ')-uniruled for some $\Lambda' \geq 0$. $\qquad\square$

3. Uniruled smooth affine varieties

We say that an affine variety A is *algebraically k-uniruled* if through every point $p \in A$ there is a polynomial map $S \to A$ passing through p where S is equal to $\mathbb{P}^1$ with at most k punctures. We want to relate this algebraic definition of uniruledness with the one in the last section. In order to do this we need to associate a Liouville domain with A.

Definition 3.1. *Let $A \subset \mathbb{C}^N$ be a smooth affine variety in $\mathbb{C}^N$. Then we define the 1-form θ_A to be equal to $\sum_{i=1}^N \frac{1}{2} r_i^2 d\vartheta_i$ restricted to A where (r_i, ϑ_i) are polar coordinates for the ith $\mathbb{C}$ factor. We have that $\omega_A := d\theta_A$ is a biholomorphic invariant [EG91]. Also for $R \gg 1$, $(B(R) \cap A, \theta_A)$ is a Liouville domain by [McL12, Lemma 2.1] where $B(R)$ is the closed ball of radius R. We will write $(\overline{A}, \theta_A)$ for such a Liouville domain and call it a* **Liouville domain associated to** *A.*

If A_1, A_2 are two isomorphic smooth affine varieties then any Liouville domain associated to A_1 is Liouville deformation equivalent to any Liouville domain associated to A_2 by Lemma 8.2 in the Appendix. The problem with the symplectic form ω_A is that it gives A infinite volume. But we need to compactify A so in order to deal with this we need another symplectic structure on A which is compatible with the compactification X of A.

Definition 3.2. *Let A be a smooth affine variety and X a smooth projective variety such that $X \setminus A$ is a smooth normal crossing divisor (an SNC compactification). Let L be an ample line bundle on X given by an effective divisor D whose support is $X \setminus A$. From now on such a line bundle will be called a* **line bundle associated to an SNC compactification X of** *A. Suppose $|\cdot|$ is some metric on L whose curvature form is a positive $(1,1)$ form. Then if s is some section of L such that $s^{-1}(0) = D$ then we define $\phi_{s,|\cdot|} := -\log|s|$ and $\theta_{s,|\cdot|} := -d^c \phi_{s,|\cdot|}$. The two form $d\theta_{s,|\cdot|}$ extends to a*

symplectic form $\omega_{|\cdot|}$ on X (which is independent of s but does depend on $|\cdot|$). We will say that $\phi_{s,|\cdot|}$ is a **plurisubharmonic function associated to** L, $\theta_{s,|\cdot|}$ *a* **Liouville form associated** L *and* $\omega_{|\cdot|}$ *a* **symplectic form on** X **associated to** L.

The aim of this section is to prove the following:

Theorem 3.3. *Let A be a smooth affine variety and $\overline{A}$ its associated Liouville domain. Then if $\overline{A}$ is (k, Λ)-uniruled then A is algebraically k-uniruled.*

We need some preliminary lemmas before we prove this theorem.

Lemma 3.4. *Let L be a line bundle associated to an SNC compactification X of A and let $|\cdot|_1$, $|\cdot|_2$ be two metrics on L whose curvature forms are positive $(1,1)$-forms. Then $(A, \omega_{|\cdot|_1})$ is symplectomorphic to $(A, \omega_{|\cdot|_2})$.*

Proof. of Lemma 3.4. We have a smooth family of symplectic forms $\omega_t := (1-t)\omega_{|\cdot|_1} + t\omega_{|\cdot|_2}$ on X. By a Moser argument we have a smooth family of symplectomorphisms $\phi_t : (X, \omega_{|\cdot|_1}) \to (X, \omega_t)$. Let $D_t := \phi_t^{-1}(D)$ where D is the associated compactification divisor for A. We have that $(X \setminus D_0, \omega_{|\cdot|_1})$ is symplectomorphic to $(X \setminus D_1, \omega_{|\cdot|_1})$ by [McL12, Lemma 5.15]. Hence $(A, \omega_{|\cdot|_1})$ is symplectomorphic to $(A, \omega_{|\cdot|_2})$. $\square$

Let (M, θ_M) be an exact symplectic manifold. We say that it is a *finite type Liouville manifold* if there is an exhausting function f (i.e. it is proper and bounded below) with the property that $df(X_{\theta_M}) > 0$ outside some compact set. Here X_{θ_M} is the $\omega_M := d\theta_M$-dual of θ_M. We say that a finite type Liouville manifold M is *strongly bounded* if there is some compact set $\mathcal{K} \subset M$ and a constant $T > 0$ so that for every point p outside $\mathcal{K}$, the time T flow of p along X_{θ_M} does not exist. In other words every point outside $\mathcal{K}$ flows to infinity within some fixed finite time.

Lemma 3.5. *Let (M, θ_M) be a strongly bounded finite type Liouville manifold and let f be its exhausting function. Choose $C \gg 1$ so that $df(X_{\theta_M}) > 0$ when $f \geq C$. Then there is a $\nu > 0$ and an exact symplectic embedding of (M, θ_M) into the Liouville domain $(f^{-1}(-\infty, C], \nu\theta_M)$. This embedding is also a homotopy equivalence.*

Proof. of Lemma 3.5. We define $M_C := f^{-1}(-\infty, C]$. Because (M, θ_M) is strongly bounded and that $df(X_{\theta_M}) > 0$ when $f \geq C$, we have a constant T so that every flowline starting at a point p with $f(p) \geq C$ flows to infinity in time less than T. Let $\phi_t : M \to M$ be the time t flow of $-X_{\theta_M}$. We define an embedding $\iota : M \hookrightarrow M_C$ by ϕ_T. The reason why ϕ_T sends M into M_C is because we know that points outside M_C flow to infinity in time less than T. We have that $e^T \iota^* \theta_M = \theta_M$. Hence ι is an exact symplectic embedding of (M, θ_M) into $(M_C, \nu\theta_M)$ where $\nu = e^T$. $\square$

Lemma 3.6. *Let L be a line bundle associated to an SNC compactification X of a smooth affine variety A and let $\theta_{s,\|\cdot\|}$ be a Liouville form associated to L. Then there is some Liouville domain (M, θ) Liouville deformation*

equivalent to $(\overline{A}, \theta_A)$ and an exact symplectic embedding $(A, \theta_{s,\|\cdot\|}) \hookrightarrow (M, \theta)$ which is also a homotopy equivalence.

Proof. of Lemma 3.6. In order to prove this Lemma we will show that $(A, \theta_{s,\|\cdot\|})$ is a strongly bounded finite type Liouville manifold and then use Lemma 3.5 to finish off the proof.

We write $D := s^{-1}(0)$ and $f_A := \phi_{s,\|\cdot\|} = -\log\|s\|$ our plurisubharmonic function associated to L. Locally around some point in D we have a holomorphic chart $z_1, \cdots, z_n$ so that $X \setminus A$ is equal to $z_1 z_2 \cdots z_k = 0$. The line bundle L is trivial over this chart and $s = z_1^{a_1} \cdots z_k^{a_k}$ with respect to this trivialization for some $a_1, \cdots, a_k > 0$. The metric $\|\cdot\|$ is equal to $e^\rho |\cdot|$ where $|\cdot|$ is the standard Euclidean metric on our trivialization of L. So $f_A = -\log\|s\| = -\rho - \sum_{i=1}^k a_i \log|z_i|$ in the above coordinate chart. We have that $\theta_A = -d^c f_A$ and $\omega_A = -dd^c f_A$. So $df_A(X_{\theta_A}) = \|df_A\|_J^2$ where $\|\cdot\|_J$ is the metric on the real cotangent bundle of X. There is some constant $\gamma > 0$ so that $\|df_A\|_J^2 \geq \gamma |df_A|^2$ where $|\cdot|$ (by abuse of notation) is the standard Euclidean metric with respect to our coordinate chart $z_1, \cdots, z_n$. This implies that:

$$df_A(X_{\theta_A}) \geq \gamma \left(-|d\rho|^2 + \sum_{i=1}^k a_k^2 \left| \frac{1}{z_i} \right|^2 \right).$$

We can assume that the functions $|z_i|$ are bounded above by some constant. This implies that for any $a > 0$, $b \in \mathbb{R}$, there is a constant $\kappa > 0$ such that if $\min(|z_i|, |z_j|) < \kappa$ then $|\frac{1}{z_i}|^2 + |\frac{1}{z_j}|^2 \geq a\log(|z_i|)\log(|z_j|) + b$. Because $f_A = -\rho - \sum_{i=1}^k a_i \log|z_i|$ we have by the previous two inequalities that

$$\tag{1} df_A(X_{\theta_A}) \geq f_A^2$$

sufficiently near D. Now lets look at a flowline $x(t)$ of X_{θ_A} near D. We have that $y(t) := f_A(x(t))$ satisfies

$$\frac{dy}{dt} \geq y^2$$

by equation (1). Solving such a differential inequality gives us:

$$y \geq \frac{y(0)}{1 - y(0)t}$$

whose solution blows up in time less than $\frac{1}{y(0)}$ (as we can assume $y(0) > 0$ because we are near D). This implies that if we are inside the region $f_A^{-1}(1, \infty)$ and also sufficiently near D then every flowline of X_{θ_A} flows off to infinity in time less than 1. Hence (A, θ_A) is a strongly bounded finite type Liouville manifold. So by Lemma 3.5 we have an exact symplectic embedding of (A, θ_A) into $(M, \theta) := (f_A^{-1}(-\infty, C], \nu\theta_A)$ where $C, \nu \gg 1$. This embedding is a homotopy equivalence.

By Lemma 8.2, $\overline{A}$ is Liouville deformation equivalent to $f_A^{-1}(-\infty, C]$ for any $C \gg 0$ which in turn is Liouville deformation equivalent to (M, θ). $\square$

The following lemma is technical and will be useful here when J is standard and also useful later on.

Lemma 3.7. *Let A be a smooth affine variety and X a smooth projective variety compactifying A. We let J be an almost complex structure on X which agrees with the standard complex structure on X near $X \setminus A$. Let $u : S \to X$ be a J holomorphic map where S is a compact nodal Riemann surface such that no component of S maps entirely in to $X \setminus A$. Let $\phi_{s, \|\cdot\|}$ be some plurisubharmonic function associated to an ample line bundle L on X. Then near $u^{-1}(X \setminus A)$ we have that $\phi_{s, \|\cdot\|} \circ u$ has no singularities.*

Proof. of Lemma 3.7. We will define $\phi := \phi_{s, \|\cdot\|} \circ u$. Near $u^{-1}(X \setminus A)$ there is a holomorphic section s of $u^* L$ whose zero set is $u^{-1}(X \setminus A)$ such that $\phi = -\log \|s\|$. Let $p \in u^{-1}(X \setminus A)$. We want to show that ϕ has no singularities near p. After trivializing $u^* L$ near p we have that $s = g(z) z^l$ where z is some coordinate function on S near $p = \{z = 0\}$ and g is a non-zero holomorphic function near p. Also $\|s\| = e^{-\psi} |s|$ where $|\cdot|$ is the standard Euclidean metric on the trivialization of L and ψ is some function. If we choose polar coordinates $z = re^{i\vartheta}$ then $\phi = \psi - l \log r - \log |g(z)|$. So $-\frac{\partial}{\partial r}(\phi)$ tends to infinity as r tends to 0 because $-\frac{\partial}{\partial r}(\psi - \log |g(z)|)$ is bounded but $\frac{\partial}{\partial r}(l \log r) = \frac{l}{r}$. Hence ϕ has no singularities near $u^{-1}(X \setminus A)$. $\square$

The next lemma shows us how to relate holomorphic curves inside smooth affine varieties with algebraic curves inside compactifications of these smooth affine varieties. This technique is a degeneration technique.

Lemma 3.8. *Suppose we have a morphism $\pi : Q \to \mathbb{C}$ of smooth quasiprojective varieties with the following properties:*

(1) *There is a symplectic form on Q compatible with the complex structure.*

(2) *The central fiber $\pi^{-1}(0)$ is equal to $F \cup E$ where F and E have the same dimension and where F is a projective variety.*

(3) *$Q \setminus E$ is isomorphic to a product $B \times \mathbb{C}$ where B is a smooth affine variety. The morphism π under the above isomorphism is the projection map $B \times \mathbb{C} \twoheadrightarrow \mathbb{C}$.*

(4) *There is a sequence $x_i \in \mathbb{C} \setminus \{0\}$ tending to zero as i tends to infinity and a holomorphic map $u_{x_i} : S_{x_i} \to \pi^{-1}(x_i)$ for each i. Here S_{x_i} is a smooth genus zero Riemann surface with $|H_1(S_{x_i}, \mathbb{Q})| \leq k - 1$. This map is not necessarily a proper map and it has energy bounded above by some constant Λ with respect to ω where Λ is independent of i.*

(5) *There is a neighbourhood N of F with the property that $u_{x_i}|_{u_{x_i}^{-1}(N)}$ is a proper map for i sufficiently large.*

(6) *All these curves u_{x_i} pass through some point $p_{x_i} \in \pi^{-1}(x_i)$ where p_{x_i} tends to some point $p \in F \setminus E$ as i tends to ∞.*

Then there is a non-trivial holomorphic curve $v : \mathbb{P}^1 \to F$ with the property that $v^{-1}(E)$ is at most k points in $\mathbb{P}^1$. Also p is contained in the image of

v. If all the curves u_{x_i} are contained inside some closed subvariety V of Q then v is also contained inside V.

Proof. of Lemma 3.8. Choose real compact codimension zero submanifolds with boundary N_1, N_2 of N with the property that the interior of N_1 contains F and the interior of N_2 contains N_1. We can perturb the boundaries of these manifolds N_1 and N_2 so that they are transverse to u_{x_i} for all i. Consider the holomorphic curves $u'_{x_i} := u_{x_i}|_{u_x^{-1}(N_2)}$. By a compactness argument [Fis11] using the manifold N_2 and curves u'_{x_i}, we have (after passing to a subsequence), a sequence of compact subcurves $\widetilde{S}_i \subset S_{x_i}$ with the following properties:

(1) the boundary of $\widetilde{S}_i$ is sent by u_{x_i} outside N_1.
(2) There is a compact surface $\widetilde{S}$ with boundary and a sequence of diffeomorphisms $a_i : \widetilde{S} \to \widetilde{S}_i$ so that $u_{x_i} \circ a_i$ C^0 converge to some continuous map $v' : \widetilde{S} \to Q$. This continuous map is smooth away from some union of curves Γ in the interior of $\widetilde{S}$ and $u'_{x_i} \circ a_i$ converge in C^∞_{loc} to v' outside Γ.
(3) The map v' is equal to $v'' \circ \phi$ where v'' is a holomorphic map from a nodal Riemann surface S with boundary to N_2 and $\phi : \widetilde{S} \to S$ is continuous surjection, and a diffeomorphism onto its image away from Γ. The curves Γ get mapped to the nodes of S under ϕ. The map v'' sends the boundary of S outside N_1.

Because each of the curves u_{x_i} are contained in $\pi^{-1}(x_i)$ and x_i tends to zero we also get that the image of v'' is contained in $\pi^{-1}(0) = F \cup E$. Also because the points p_{x_i} converge to $p \in F$, we have some $s \in S$ such that $v''(s) = p$. Because v'' sends the boundary of S outside N_1 and that the interior of N_1 contains F, we have an irreducible component $\mathbb{P}^1 = K$ inside our nodal Riemann surface S where v'' maps K to F. Also we can assume that $s \in K$. Our map v is defined as v'' restricted to $K = \mathbb{P}^1$.

We now wish to show that $v^{-1}(E)$ is at most k points. There is a natural exhausting plurisubharmonic function ρ on B which we can construct using Definition 3.1. We pull back ρ to $\widetilde{\rho}$ under the natural projection map $P_B : B \times \mathbb{C} \twoheadrightarrow B$. Here we identify $B \times \mathbb{C}$ with $Q \setminus E$. We know that $v^{-1}(E)$ is the disjoint union of l points for some l. We want to show that $l \leq k$. The image $v(K)$ of v is a one dimensional projective subvariety of F, and $v(K) \setminus E$ is a (possibly singular) affine subvariety of B. By Lemma 3.7 we have that ρ restricted to $v(K)$ is non-singular outside a compact set. Hence for $C \gg 1$ we have that $\rho^{-1}(C)$ is transverse to v and that $(\rho \circ v)^{-1}(C)$ is a disjoint union l circles. Also by making C large enough, we have that $v^{-1}(\rho^{-1}(-\infty, C])$ is connected. Because the maps $u_{x_i} \circ a_i$ converge in C^∞_{loc} to v near these l circles for i large enough and that these maps C^0 converge, we get that the connected component S'_i of $(\rho \circ u_{x_i})^{-1}((-\infty, C])$ passing through p_x has l boundary components for $i \gg 1$. Because

(1) each curve u_{x_i} maps to a smooth affine variety $\pi^{-1}(x_i)$ and

(2) $(\rho \circ u_{x_i})^{-1}((-\infty, C])$ is compact for i large enough and
(3) $|H^1(S_{x_i}, \mathbb{Q})| \le k - 1$,

we have by Lemma 2.3 that $H_1(S_i', \mathbb{Q})$ has rank at most $k - 1$ which implies that S_i' has at most k boundary circles for $i \gg 1$. But we know for i sufficiently large that it is also has l boundary components, hence $l \le k$. This implies that $v^{-1}(E)$ is a union of $\le k$ points and passes through p.

Now suppose that all of these curves u_{x_i} are contained inside some closed subvariety V. Because $u_{x_i} \circ a_i$ C^0 converges to v' we have that v' is a subset of V because V is a closed subset of Q. The image of v is contained inside the image of v' which implies that the image of v is contained inside V. $\square$

Proof. of Theorem 3.3. Let X' be some compactification of A by a projective variety. By the Hironaka resolution of singularities theorem [Hir64] we can resolve the compactification X' of A so that it is some smooth projective variety X with the property that $X \setminus A$ is a smooth normal crossing divisor. This variety can be embedded into $\mathbb{P}^N$ so that $X \setminus A$ is equal to X intersected with some linear hypersurface $\mathbb{P}^{N-1}$ in $\mathbb{P}^N$. So we can view A as a subvariety of $\mathbb{C}^N = \mathbb{P}^N \setminus \mathbb{P}^{N-1}$. We will let D_X be the effective ample divisor given by restricting $\mathbb{P}^{N-1}$ to X.

We start with $\mathbb{P}^1 \times \mathbb{P}^N$. The divisor $D := \{\infty\} \times \mathbb{P}^N + \mathbb{P}^1 \times \mathbb{P}^{N-1}$ is ample. Let $P := \mathrm{Bl}_{\{0\} \times \mathbb{P}^{N-1}} \mathbb{P}^1 \times \mathbb{P}^N$ be the natural blowup map along $\{0\} \times \mathbb{P}^{N-1}$ and let $\widetilde{D}$ be the proper transform of D in P. We let E be the exceptional divisor. Then $k\widetilde{D} + (k-1)E$ is ample inside P for $k \gg 1$. Let $\pi : P \to \mathbb{P}^1$ be the composition of the blowdown map with the projection map to $\mathbb{P}^1$. The fiber $\pi^{-1}(0)$ is a union of two divisors $F + E$ and this is linearly equivalent to $\pi^{-1}(\infty)$. Hence E is linearly equivalent to $\pi^{-1}(\infty) - F$. Let D' be the divisor $k\widetilde{D} + (k-1)(\pi^{-1}(\infty) - F)$. This is ample and the associated line bundle $L_{D'}$ admits a metric $\|\cdot\|$ whose curvature form is a positive $(1,1)$-form. This gives us a symplectic form on X. Let s be a meromorphic section of $L_{D'}$ so that $s^{-1}(0) - s^{-1}(\infty) = D'$. We have that $-d^c \log \|s\|$ restricted to $\pi^{-1}(x) \setminus \mathrm{support}(D')$ $(x \ne 0)$ makes this fiber into a Liouville manifold. We have that D' is the disjoint union of $D_1' := k\widetilde{D} + (k-1)\pi^{-1}(\infty)$ and $-(k-1)F$. Also $-\log \|s\|$ tends to $+\infty$ as we approach D_1' and $-\infty$ as we approach F. Hence $P_C := \left((-\log \|s\|)^{-1}((-\infty, C]) \right) \cup F$ is a compact submanifold of $X \setminus \mathrm{support}(D_1')$ for generic $C \gg 1$ whose interior contains F.

Consider $\mathbb{P}^1 \times X \subset \mathbb{P}^1 \times \mathbb{P}^N$ and let P_X be the proper transform of $\mathbb{P}^1 \times X$ inside P. We let π_X be the restriction of π to P_X. We have $A_x := \pi_X^{-1}(x) \setminus \mathrm{support}(D_1')$ are all isomorphic smooth affine varieties when $x \ne 0$. Also if $X_x := \pi_X^{-1}(x)$ then these isomorphisms extend to isomorphisms $\phi_{x,y} : X_x \to X_y$ so that $\phi^* L|_{X_y} = L|_{X_x}$. All these affine varieties are isomorphic to A. So by Lemma 3.4 we have that all these affine varieties are symplectomorphic with respect to the symplectic form $-dd^c \log \|s\|$. Combining this with Lemma 3.6 we then get that all these varieties can be

codimension 0 symplectically embedded into a fixed Liouville domain (M, θ) which is Liouville deformation equivalent to $\overline{A}$. Also these embeddings are homotopy equivalences. Because $\overline{A}$ is (k, Λ) uniruled, we have by Lemma 2.6 that (M, θ) is (k, Λ') uniruled for some $\Lambda' > 0$. We define $P_A \subset P_X$ to be equal to $P_X \setminus (\text{support}(D_1') \cup E)$. This is isomorphic to $\mathbb{C} \times A$. Let p be any point in $P_X \cap (F \setminus E)$ and let $x_i \in \mathbb{C} \setminus \{0\}$, $p_{x_i} \in P_C \cap A_{x_i}$ be a family of points in A_{x_i} which all converge to p as i tends to ∞. For every x_i choose a Liouville domain N_{x_i} which is an exact codimension 0 symplectic submanifold of A_{x_i} containing $P_C \cap A_{x_i}$ and so that the embedding map $N_{x_i} \hookrightarrow A_{x_i}$ is a homotopy equivalence. By Lemma 2.2, we have that N_{x_i} is (k, Λ') uniruled because it can be symplectically embedded into M so that the embedding is a homotopy equivalence. So for each i there is a proper J holomorphic curve $u_{x_i} : S_{x_i} \to N_{x_i}^0$ where u_{x_i} has energy $\leq \Lambda'$ and $|H_1(S_{x_i}, \mathbb{Q})| \leq k - 1$. In particular $u_{x_i}|_{u_x^{-1}(P_C^o)}$ are properly embedded holomorphic curves inside the interior P_C^o of the compact manifold P_C. By Lemma 3.8, there is an algebraic map $v : \mathbb{P}^1 \to P_X \cap F$ with the property that $v(q) = p$ for some $q \in \mathbb{P}^1$ and $v^{-1}(E)$ is a union of at most k points. After identifying A with $P_X \cap (F \setminus E)$ we then get that $v|_{v^{-1}(A)}$ is also algebraic and passing through p and $v^{-1}(A)$ is $\mathbb{P}^1$ with at most k punctures. Hence A is algebraically k-uniruled. $\qquad\square$

4. Introduction to Gromov Witten invariants

Genus 0 Gromov Witten invariants for general symplectic manifolds have now been defined in many different ways: [FO99], [CM07], [Hof11] and [LT98b]. Earlier work for special symplectic manifolds such as projective varieties of complex dimension 3 or less are done in [Rua96], [Rua94] and [RT95]. Many of the applications of this paper appear in complex dimension 3 or less. These invariants can also be defined in a purely algebraic way [LT98a], [BF97] and [Beh97] but we will not use these theories here. We will use the Gromov Witten invariants defined for general symplectic manifolds. All of our calculations are done for complex structures where all of the curves in the relevant homology class are regular and unobstructed (and also somewhere injective) and so are relatively easy calculations. Also most of these (or similar) calculations have been done before in [McD90], [Rua99] and [Kol98].

We start with a compact symplectic manifold X, a natural number k and an element $\beta \in H_2(X)$. Let $d := 2(n - 3 + k + c_1(X).\beta)$ where n is half the dimension of X. Choose k cohomology classes $\alpha_1, \cdots, \alpha_k \in H^*(X, \mathbb{Q})$ so that the sum of their degrees is d. For any compatible almost complex structure one has the set $\mathcal{M}(\beta, J, k)$ of J holomorphic maps $u : S \to X$ where S is a genus 0 compact nodal Riemann surface with k labeled marked points. This nodal curve has to be stable which means that if an irreducible component of this surface maps to a point then that component must have at least three of these marked points. There are natural maps

$\mathrm{ev}_i : \mathcal{M}(\beta, J, k) \to X$ which send a curve $u : S \to X$ to $u(x_i)$ where x_i is the ith marked point in S. It turns out (in nice circumstances) that $\mathcal{M}(\beta, J, k)$ is a topological space with a homology class

$$[\mathcal{M}(\beta, J, k)]^{\mathrm{vir}} \in H_d(\mathcal{M}(\beta, J, k), \mathbb{Q}).$$

One then has

$$\langle \alpha_1, \cdots, \alpha_k \rangle_{0,\beta}^X := \int_{[\mathcal{M}(\beta, J, k)]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^* \alpha_k.$$

The genus 0 Gromov Witten invariant

$$\langle \alpha_1, \cdots, \alpha_k \rangle_{0,\beta}^X \in \mathbb{Q}$$

satisfies the following properties:

(1) If $\langle \alpha_1, \cdots, \alpha_k \rangle_{0,\beta}^X \neq 0$ for some $\alpha_1, \cdots, \alpha_k$ then for every compatible J, there exists a J holomorphic map $u : S \to X$ from a genus 0 nodal curve S representing the class β.

(2) Suppose that X is a smooth projective variety with its natural complex structure J. Suppose that every rational curve C representing the class β is smooth, embedded, and satisfies $H^1(C, T_X|_C) = 0$ where T_X is the tangent sheaf. Then $\langle \alpha_1, \cdots, \alpha_k \rangle_{0,\beta}^X \neq 0$ for some $\alpha_1, \cdots, \alpha_k$.

The reason why (2) is true is that $\mathcal{M}(\beta, J, k)$ in this case is a complex manifold of dimension d for every k and $[\mathcal{M}(\beta, J, k)]^{\mathrm{vir}}$ is equal to its fundamental class. For k large enough, the map:

$$\mathrm{ev}_1 \times \cdots \times \mathrm{ev}_k : \mathcal{M}(\beta, J, k) \to X^k$$

is a holomorphic map which is a branched cover onto its image. If we restrict the natural product symplectic structure $\omega_{X^k} := \omega_1 + \cdots + \omega_k$ on X^k to $\mathcal{M}(\beta, J, k)$ then it is also a symplectic structure on this moduli space away from the branching locus. Hence $\omega_{X^k}^d$ restricted to $\mathcal{M}(\beta, J, k)$ is a positive multiple of the volume form on an open dense subset and so it evaluates non-trivially with the fundamental class. In particular we have that $\omega_1^{i_1} \wedge \cdots \wedge \omega_k^{i_k}$ evaluates non-trivially with the fundamental class for some $i_1, \cdots, i_k$. So if we choose $\alpha_l := \omega_1^{i_l}$ then $\langle \alpha_1, \cdots, \alpha_k \rangle_{0,\beta}^X \neq 0$. This argument is almost exactly the same as an argument at the end of the proof of [HLR08, Theorem 4.10].

5. Uniruledness criteria for affine varieties

In this section we will give another definition of uniruledness for smooth affine varieties. The main theorem of this section is to show that any smooth affine variety satisfying this uniruledness condition has an associated Liouville domain which is also (k, Λ)-uniruled.

We say that a smooth affine variety A is **compactified k-uniruled** if A has some compactification X by a smooth projective variety so that if $D = X \setminus A$ then we have the following properties:

(1) There is an effective ample divisor D_X on X whose support is D and whose associated line bundle has a metric whose curvature form gives us some symplectic form ω_X on X.

(2) Let J be an almost complex structure compatible with ω_X which is the standard complex structure near D. Then there is a dense set $U_J \subset A$ so that for every point $p \in U_J$ such that J is integrable near p we have a J holomorphic map $u : \mathbb{P}^1 \to X$ passing through p such that $u^{-1}(D \setminus A)$ is a union of at most k points.

(3) The energy of this curve is bounded above by some fixed constant Λ.

We will now give some easier criteria for being compactified k-uniruled. Our symplectic form on X comes from some ample divisor.

Lemma 5.1. *Suppose that we have a morphism $\pi : X \to B$ whose generic fiber is $\mathbb{P}^1$ where the base B is projective. Let $\beta \in H_2(X)$ be the class of this curve. Then for every compatible almost complex structure J which is integrable on some open set U containing a point p, there is some J holomorphic curve $u : S \to X$ passing through p representing the class of the fiber. Here S is a genus 0 nodal curve.*

Proof. of Lemma 5.1. Let F be any regular fiber of π. This is isomorphic to $\mathbb{P}^1$. Blow up X to $\widetilde{X}$ at some point in F. Let $\widetilde{F}$ be the proper transform of F inside X and let $\widetilde{\beta} \in H_2(\widetilde{X}, \mathbb{Q})$ be its respective homology class. The only curve in this homology class is $\widetilde{F}$. If we restrict the tangent bundle $T_{\widetilde{X}}$ to this curve then it is isomorphic to: $\mathcal{O}(2) \oplus \mathcal{O}(-1)^{\oplus n-1}$. Hence $H^1(\widetilde{F}, T_X|_{\widetilde{F}}) = 0$. By property (2) there exists $\alpha_1, \cdots, \alpha_k$ such that $\langle \alpha_1, \cdots, \alpha_k \rangle^X_{0, \widetilde{\beta}} \neq 0$.

Let p_i be a sequence of points in U converging to p where p_i is contained inside a smooth fiber F_i. Because J is integrable in U, we can blowup X at p_i giving us a new symplectic manifold X_i along with a compatible almost complex structure so that the blowdown map is holomorphic. Let $\widetilde{F}_i$ be the proper transform of F_i in X_i and $\beta_i \in H_2(X_i, \mathbb{Z})$ its respective homology class. Then because $\langle \alpha_1, \cdots, \alpha_k \rangle^X_{0, \beta_i} \neq 0$ for some cohomology classes α_i, we have by property (1) a J holomorphic curve $u'_i : S_i \to X_i$ representing β_i. By composing this map with the blowdown map, we get a J holomorphic curve $u_i : S_i \to X$ passing through p_i and representing β. By a Gromov compactness argument one then gets a holomorphic curve $u : S_i \to X$ passing through p representing the class β. $\qquad\square$

Lemma 5.2. *Suppose that we have a morphism $\pi : X \to B$ whose generic fiber is $\mathbb{P}^1$ where B is a projective variety. Suppose also that D' is an effective nef divisor whose support is equal to D with the property that $\beta.D' \leq k$. Then A is compactified k-uniruled.*

Proof. of Lemma 5.2. Choose any effective ample divisor D_X whose support is D and let ω_X be the symplectic form associated to this divisor. Let J be any compatible almost complex structure which is standard near D.

Let p be any point in A where J is integrable near p. By Lemma 5.1, there is a J holomorphic curve $u : S \to X$ representing the homology class of the fiber passing through p. Here S is a nodal curve with irreducible components $S_1, \cdots, S_l$. Let S_i be any component passing through p. Then by positivity of intersection we have that $u(S_i).D \le u(S_i).D'$ and $u(S_i).D' \le \sum_i u(S_i).D' = u(S).D' \le k$. Hence $u(S_i).D \le k$. The energy of the curve $u|_{S_i}$ is bounded above by $\beta.D_X$. Because S_i is isomorphic to $\mathbb{P}^1$ we then get that A is compactified k-uniruled. $\qquad\square$

Theorem 5.3. *Suppose that A is a smooth affine variety that is compactified k-uniruled. Let $\overline{A}$ be its associated Liouville domain. Then $\overline{A}$ is (k, Λ)-uniruled for some Λ.*

Before we prove this theorem we need a lemma and a definition. Let X be a smooth projective variety with a smooth normal crossing divisor D so that $X \setminus D$ is affine. A map $u : S \to X$ is said to be a *k-curve* if every irreducible component Σ of S either maps to D, or $u^{-1}(D) \cap \Sigma$ is a finite set of size at most k.

Lemma 5.4. *Let A be a smooth affine variety and X a smooth projective variety compactifying A. We equip X with a symplectic form $\omega_{\|\cdot\|}$ coming from some ample line bundle. We let J be a compatible almost complex structure on X which agrees with the standard complex structure on X near $X \setminus A$. Let $u_i : S_i \to X$ be a sequence of J holomorphic maps where S_i is a connected genus 0 nodal Riemann surface and where all the u_i's have energy bounded above by some fixed constant. If the u_i are all k curves and Gromov converge to $u : S \to X$ then u is also a k curve.*

Proof. of Lemma 5.4. Let $S_i^o := u_i^{-1}(A)$ and $S^o := u^{-1}(A)$. We want to show that the rank of H_1 of each connected component of S^o is at most $k - 1$. Because the u_i Gromov converge, that means that there is a smooth real surface $\widetilde{S}$ and a series of continuous maps $\alpha_i : \widetilde{S} \to S_i$, $\alpha : \widetilde{S} \to S$ satisfying:

(1) α_i and α are diffeomorphisms away from a 1-dimensional submanifold $\Gamma \subset \widetilde{S}$ and away from the nodes of S_i and S.
(2) Γ maps to the nodes of S under α_0.
(3) $u_i \circ \alpha_i$ C^0 converge to $\alpha_0 \circ u$ and these maps C^∞_{loc} converge away from Γ.

Choose an exhausting plurisubharmonic function $\phi : A \to \mathbb{R}$ associated to some line bundle L, section s and metric $\|\cdot\|$ on L. Gromov convergence means that for $c \gg 1$ and $i \gg 1$ we have that every node of S_i^o and also $\alpha_i(\Gamma) \cap S_i^o$ is mapped via u_i to $\phi^{-1}(-\infty, c]$. We can also assume that the same is true for $S^0 := u^{-1}(A)$. For large enough i and for generic c large enough we have u_i is smooth near $\phi^{-1}(c)$ and also transverse to this hypersurface. We can assume the same properties hold for u. Let Σ_i be a sequence of connected components of S_i^o which converge to a connected component Σ

of S^o. We have that $\Sigma_i \cap u_i^{-1}(\phi^{-1}(c))$ is a union of l_i smooth circles in S_i and $\Sigma \cap u^{-1}(\phi^{-1}(c))$ is a union of l circles for some l_i, l. This means that $H_1(\Sigma_i \cap u_i^{-1}(\phi^{-1}(-\infty, c)))$ has rank $\leq l_i - 1$. By Lemma 2.3 we then get that $H_1(\Sigma \cap u_i^{-1}(\phi^{-1}(-\infty, c)))$ has rank less than or equal to $|H_1(\Sigma_i)| \leq k - 1$. Hence $l_i \leq k - 1$ for all i. So $\Sigma_i \cap u_i^{-1}(\phi^{-1}(c))$ is a union of at most k circles. Because $u_i \circ \alpha_i$ C^∞ converge to $u_i \circ \alpha$ near $\phi^{-1}(c)$ we get that $\Sigma \cap u^{-1}(\phi^{-1}(c))$ is also a union of at most k circles. This is true for all c sufficiently large. Hence $\text{rank}(H_1(\Sigma)) \leq k - 1$ for each connected component Σ of S^o. Hence u is a k curve. $\qquad\square$

Proof. of Theorem 5.3. Because A is compactified k-uniruled, we have a compactification X with divisor D so that:

(1) There is an effective ample divisor D_X on X with support D whose associated line bundle has a metric with curvature form ω_X on X. Here ω_X is a symplectic form.
(2) Let J be an almost complex structure compatible with ω_X which is the standard complex structure near D. Then there is a dense set $U_J \subset A$ so that for every point $p \in U_J$ such that J is integrable near p we have a J holomorphic map $u : \mathbb{P}^1 \to X$ passing through p which is a k curve.
(3) The energy of this curve is bounded above by some fixed constant Λ'.

We have a plurisubharmonic function $\rho := -\log|s|$ where s is a section of L with $s^{-1}(0) = D_X$. For $c \gg 1$ we have that $A_c := \rho^{-1}(-\infty, c]$ is a Liouville domain deformation equivalent to $\overline{A}$ by Theorem 8.2. We now let J be any almost complex structure which coincides with the standard one near D and coincides with any convex almost complex structure inside A_c. Let p be any point in the interior of A_c where J is integrable on a neighbourhood of p. Choose a sequence of points $p_i \in U_J$ converging to p. There is a map $u_i : \mathbb{P}^1 \to X$ of energy bounded above by Λ' passing through p_i so that u_i is a k curve. There is a subsequence which Gromov converges to a map $v : S \to X$ of energy bounded above by Λ' passing through p. Here S is a genus 0 nodal curve. By Lemma 5.4, we then get that v is a k curve. Let S' be an irreducible component of S whose image under v contains p, $S'' := S' \cap v^{-1}(A)$ and $\Sigma := S'' \cap v^{-1}(A_c^0)$ where A_c^0 is the interior of A_c. By Lemma 2.3 we have that $|H_1(\Sigma, \mathbb{Q})| \leq k$ because $|H_1(S'', \mathbb{Q})| \leq k$. Let $u := v|_\Sigma$. The energy of u is bounded above by Λ'. This implies that A_c is (k, Λ')-uniruled. By Corollary 2.6 we then get that $\overline{A}$ is (k, Λ)-uniruled for some $\Lambda > 0$. $\qquad\square$

6. Log Kodaira dimension and uniruledness

We will now define log Kodaira dimension. Let L be any line bundle on a projective variety X. If $L^{\otimes k}$ has no global sections for any k then we define $\kappa(L) := -\infty$. Otherwise $L^{\otimes k}$ defines a rational map from X

to $\mathbb{P}(H^0(L^{\otimes k}))$ for some k. We define $\kappa(L)$ in this case to be maximum dimension of the image of this map over all k where this map is defined. The number $\kappa(L)$ is called the *Kodaira dimension of* L. If Q is any smooth quasiprojective variety then we define its *log Kodaira dimension* $\overline{\kappa}(Q)$ as follows: Choose some compactification of Q by a smooth projective variety X so that the associated compactification divisor D is smooth normal crossing. The *log Kodaira dimension* of Q is defined to be $\kappa(K_X + Q)$ where K_X is the canonical bundle of X. This an invariant of Q up to algebraic isomorphism.

Before we look at smooth affine varieties in dimension 2 and 3 we need a lemma relating uniruledness with log Kodaira dimension.

Lemma 6.1. *Suppose that A is algebraically k-uniruled. If $k = 1$ then A has log Kodaira dimension $-\infty$ and if $k = 2$ then A has log Kodaira dimension $\leq dim_{\mathbb{C}} A - 1$.*

Proof. of Lemma 6.1. First of all, we compactify A to some smooth projective variety X. Let D be the compactification divisor. Because A is algebraically k-uniruled, we have that X is uniruled by $\mathbb{P}^1$'s. By using the theory of Hilbert schemes (see [Kol96]) there is a surjective morphism:

$$\mathrm{ev} : M \times \mathbb{P}^1 \twoheadrightarrow X$$

where M is a reduced projective variety. We define $D_M := \mathrm{ev}^{-1}(D)$. We let V be the subvariety of M with the property that $q \in M$ is contained in V if and only if $D_M \cap (\{q\} \times \mathbb{P}^1)$ is a set of size at most k. Because A is k uniruled we can assume that M satisfies: $\mathrm{ev}(V \times \mathbb{P}^1)$ is dense in X . Hence we have a dominant morphism $(V \times \mathbb{P}^1) \setminus D_M \twoheadrightarrow A$. We will define W be equal to $(V^{\mathrm{sm}} \times \mathbb{P}^1) \setminus D_M$ where V^{sm} is the smooth part of V which is a non-empty Zariski open subset of V. In particular we have a morphism π_W from W to A whose image contains a dense open set. We can choose $V' \subset V$ to be a subvariety of complex dimension $\dim(X) - 1$ so that the image $\pi_W((V' \times \mathbb{P}^1) \setminus D_M)$ still contains a dense open subset of A. We define W' to be $(V' \times \mathbb{P}^1) \setminus D_M$. So $\pi_{W'} := \pi_W|_{W'}$ is a dominant morphism from W' to A. The projection map $W' \twoheadrightarrow V'$ has generic fiber equal to $\mathbb{P}^1$ minus at most k points. By the Iitaka Easy Addition Theorem ([Iit77, Theorem 4], [Iit82, Theorem 11.9]) we have that the log Kodaira dimension of W' is equal to $-\infty$ if $k = 1$ and it is $\leq \dim_{\mathbb{C}}(A) - 1$ if $k = 2$. Because there is a dominant morphism from W' to A, we have by the logarithmic ramification formula ([Iit77], [Iit82, Theorem 11.3]) that the log Kodaira dimension of W' is greater than or equal to the log Kodaira dimension of A. Combining the above two facts we have that if $k = 1$ then A has log Kodaira dimension $-\infty$ and if $k = 2$ then A has log Kodaira dimension $\leq \dim_{\mathbb{C}}(A) - 1$. $\square$

Lemma 6.2. *Suppose that A and B are symplectomorphic smooth affine varieties. Suppose that A is compactified k-uniruled. If $k = 1$ then B has log Kodaira dimension $-\infty$ and if $k = 2$ then B has log Kodaira dimension $\leq dim_{\mathbb{C}} A - 1$.*

Proof. of Lemma 6.2. By Theorem 5.3 we get that the Liouville domain $\overline{A}$ associated to A is (k, Λ) uniruled. Because B is symplectomorphic to A we then get by Lemma 2.5 that the Liouville domain $\overline{B}$ is (k, Λ') uniruled. So by Theorem 3.3, B is algebraically k uniruled. Hence by Lemma 6.1, B has the required log Kodaira dimension. $\qquad\square$

6.1. **Dimension 2.** The aim of this section is to prove:

Theorem 6.3. *Let A, B be symplectomorphic acyclic smooth affine surfaces. Then they have the same log Kodaira dimension.*

Before we prove this we need a compactified uniruled criterion in dimension 2 and some other preliminary lemmas.

Lemma 6.4. *Let X be any compact symplectic manifold of real dimension 4 and J any almost complex structure compatible with the symplectic form. Then there is a dense subset of points $U_J \subset X$ with the property that every J holomorphic map $u : \mathbb{P}^1 \to X$ with $u(\mathbb{P}^1) \cap U_J \neq \varnothing$ satisfies $u_*([\mathbb{P}^1])^2 \geq 0$. In fact U_J is a countably infinite intersection of open dense subsets.*

Proof. of Lemma 6.4. Let E be a homology class satisfying $E.E < 0$. Let $u_i : \mathbb{P}^1 \to X$, $i = 1, 2$ be two J holomorphic curves representing this class. We have that $(u_1)_*([\mathbb{P}^1]) \cdot (u_2)_*([\mathbb{P}^1])$ is negative. By positivity of intersection we then have that the images of u_1 and u_2 must coincide. We write E_J for this image. By Sard's theorem, the complement of this image is open and dense. The set of images of J holomorphic curves $u : \mathbb{P}^1 \to X$ with negative self intersection number is $\cup_{E \in H_2, E.E < 0} E_J$. The complement is a countable intersection of open dense subsets, which is also dense. $\qquad\square$

Lemma 6.5. *Suppose we have a morphism $\pi : X \to B$ where X is a smooth projective surface and B is a curve. Let ω_X be a symplectic form associated to an effective ample divisor D_X and J a compatible almost complex structure. Suppose that we have a J holomorphic map $v : \Sigma \to X$ whose fundamental class represents the fiber $[F] \in H_2(X)$, and with the property that every irreducible component Σ' of Σ satisfies $v_*([\Sigma']).[F] = 0$. Then there is a dense set $U_J \subset X$ with the property that every J holomorphic map $u : S \to X$ where S is a connected nodal curve which intersects U_J and represents $[F]$ has the property that S is irreducible.*

Proof. of Lemma 6.5. We choose U_J to be the set of points with the property that every J holomorphic curve passing through a point in U_J has some irreducible component with non-negative intersection number. This is dense by Lemma 6.4. Let S be a union of irreducible components $S_1, \cdots, S_l$. We will suppose without loss of generality that $S_1 \cap U_J$ is non-empty, and so that $(u|_{S_1})^2 \geq 0$. Suppose for a contradiction, $u_*([S_i]).[F] < 0$ for some i. Then by positivity of intersection we have that $u(S_i)$ is contained inside $v(\Sigma')$ for some irreducible component Σ' of Σ. Because $v_*([\Sigma']).[F] = 0$, we have $u_*([S_i]).[F] = 0$ which is a contradiction. Hence $u_*([S_i]).[F] \geq 0$ for

each i. Because $\sum_i u_*([S_i]).[F] = 0$, this implies that $u_*([S_i]).[F] = 0$ for all i.

Suppose for a contradiction, S has more than one irreducible component. Because S is connected we have then that $u_*([S_1]).u_*([S_j]) \neq 0$ for some $j \neq 1$, and because $(u_*([S_1]))^2 \geq 0$ we then get that $u_*([S_1]).(\sum_i u_*([S_i])) > 0$ which is impossible because $u_*([S]) = [F]$. Hence S is irreducible. $\square$

Corollary 6.6. *Let A be a smooth affine variety with an SNC compactification X, and let D be the associated compactification divisor. Suppose we have a morphism $\pi : X \to B$ satisfying the hypotheses of Lemma 6.5 for any compatible J which is standard near D. Suppose that a general fiber of π intersects D k times. Then A is compactified k uniruled.*

Proof. of Corollary 6.6. By Lemma 6.5, there is a dense subset $U_J \subset A$ with the property that any J holomorphic curve $u : S \to X$ representing F passing through $p \in U_J$ has the property that S is irreducible. Let p be any point in U_J such that J is integrable near p, then by Lemma 5.1, we have that there is such a J holomorphic map passing through p. Because S is irreducible, it intersects D in at most k points. Putting all of this together gives us that A is compactified k uniruled. $\square$

Lemma 6.7. *Suppose that X is a smooth projective surface and D, E divisors so that:*

   (1) E is irreducible and $D \cup E$ is smooth normal crossing.
   (2) If D' is the union of divisors in D not intersecting E then D' is connected and intersects every irreducible component of D.
   (3) There is an effective nef divisor G whose support is contained in D'.

Then there is an effective nef divisor D_X whose support is D' so that:

   (1) For every irreducible curve C in D and not in the support of G, we have $D_X.C > 0$.
   (2) $D_X.E = 0$.

Proof. of Lemma 6.7. Suppose that W is any effective nef divisor whose support is in D' and contains support(G). Also suppose every irreducible curve C inside support(W) but not in support(G) satisfies $C.W > 0$. Let C be any irreducible curve of D' not contained in the support of W. Because D' is connected we can assume that $C.W \neq 0$. We let $W' := \kappa W + C$ for $\kappa \gg 1$. This is an effective nef divisor with larger support than W. For κ large enough we have $C.W' > 0$. Hence every irreducible curve C inside support(W') satisfies $C.W' > 0$ if C is not in support(G). Therefore we can construct effective nef divisors starting with G with larger and larger support until we get an effective nef divisor D_X whose support is equal to D'. Every irreducible curve C in D' intersects D_X positively unless it is in the support of G. Also if C is an irreducible curve in D not contained in D' then it intersects D' and hence $C.D_X > 0$. We also have that $D_X.E = 0$. $\square$

Lemma 6.8. *Let $A = X \setminus D$ be a smooth affine surface where X is a smooth projective variety and D is a connected smooth normal crossing divisor. Suppose that we have a morphism $\pi : X \to B$ with the following properties:*

(1) *The generic fiber is $\mathbb{P}^1$. The base B is a smooth projective curve.*
(2) *If F is a fiber then $F.D \leq k$ for some k.*
(3) *There are two different points $b_1, b_2 \in B$ with the following property: $\pi^{-1}(b_i) = E_i \cup F_i$ (as reduced curves) where E_i is an irreducible smooth curve satisfying $E_i.E_i = -1$, and F_i is reduced.*
(4) *$F_i \subset D$ but E_i is not contained in D. Also $D \cup E_1 \cup E_2$ is a smooth normal crossing divisor.*
(5) *$E_2.D = E_2.F_2$.*
(6) *There is an effective nef divisor G with the property that $E_i.G = 0$ and whose support is contained inside D.*
(7) *The union of irreducible components of D not containing E_i is connected.*

Then A is compactified k uniruled.

Proof. of Lemma 6.8. Choose any compatible symplectic structure ω_X coming from an effective ample divisor D_X whose support is D. Let J be any compatible almost complex structure which is standard near D. We will complete this proof in 3 steps. In Step 1 we will construct J holomorphic curves representing $[E_i]$ such that no irreducible component is contained in D. In Step 2 we will show that each irreducible component of one the curves from Step 1 has intersection number zero with the fiber. In Step 3 we will construct our J holomorphic curve passing through p and intersecting D at most k times using Corollary 6.6.

Step 1: By [McD90, Lemma 3.1] there is a J holomorphic map $u_i : \Sigma_i \to X$ from a connected genus 0 nodal Riemann surface Σ_i representing the exceptional class $[E_i] \in H_2(X)$. We assume that no irreducible component of Σ_i maps to a point. Let $\Sigma_i^1, \cdots, \Sigma_i^{l_i}$ be the irreducible components of Σ_i. We will now show that $u_i(\Sigma_i^j)$ is not contained in D for each i, j.

By using properties (4),(7) and (6) combined with Lemma 6.7 we have an effective nef divisor D_i' satisfying:

(a) For every irreducible curve C in D and not in the support G, we have $D_i'.C > 0$.
(b) $D_i'.E_i = 0$.

Suppose for a contradiction that $(u_i)_*([\Sigma_i^y]) \subset D$ for some y. Because $[E_i]^2$ is negative and the intersection product of E_i with any irreducible component of D_i' is non-negative we have that E_i cannot be represented by an effective divisor whose support is in D_i'. If $(u_i)_*([\Sigma_i^y]) \subset D_i'$ then the previous fact tells us that there is some Σ_i^x satisfying $(u_i)_*([\Sigma_i^x]).D_i' \neq 0$. But this is impossible because D_i' is nef and $E_i.D_i' = 0$. Hence Σ_i^y is not contained in D_i', so by property (a), $(u_i)_*([\Sigma_i^y]).D_i' \neq 0$. This is impossible as D_i' is nef

and has intersection number 0 with E_i. Hence $u_i(\Sigma_i^x)$ is not contained in D for all i, x.

Step 2: The aim in this step is to show that if $[F]$ is the class of a fiber of π then $(u_1)_*([\Sigma_1^i]).[F] = 0$ for all i. Suppose for a contradiction that $(u_1)_*([\Sigma_1^1]).[F] \neq 0$. Then because $[E_1].[F] = 0$ we have $(u_1)_*([\Sigma_1^1]).[F] = -\sum_{j=2}^{l_1}(u_1)_*([\Sigma_1^j]).[F]$. So without loss of generality we can assume that $(u_1)_*([\Sigma_1^1]).[F] < 0$. We can represent $[F]$ by $[D_{F_2}] + \kappa(u_2)_*([\Sigma_2])$ by property (3) where D_{F_2} is an effective divisor whose support is exactly F_2 and κ is a positive integer. Because $(u_1)_*([\Sigma_1^1])$ does not map to D, we have by positivity of intersection that $(u_1)_*([\Sigma_1^1]).[D_{F_2}] \geq 0$. Hence $(u_1)_*([\Sigma_1^1]).(u_2)_*([\Sigma_2]) < 0$ because $\kappa > 0$. By positivity of intersection this means that $u_1(\Sigma_1^1) \subset u_2(\Sigma_2^l)$ for some l. Without loss of generality we will assume that $l = 1$.

We have that $E_2.F_2 = E_2.D$ by property (5) and that $(u_2)_*([\Sigma_2^i]).[D] \geq 1$ because A is an exact symplectic manifold. Because $(u_i)_*([\Sigma_i^j])$ is not contained inside D for all i, j, we have $(u_i)_*([\Sigma_i^j]).[F_2] \leq (u_i)_*([\Sigma_i^j]).[D]$. Using the above two facts,

$$(u_2)_*([\Sigma_2^1]).[F_2] =$$

$$\sum_{j=1}^{l_2}(u_2)_*([\Sigma_2^j]).[D] - \sum_{j=2}^{l_2}(u_2)_*([\Sigma_2^j]).[F_2] \geq (u_2)_*([\Sigma_2^1]).[D] > 0.$$

But this means that $(u_1)_*([\Sigma_1^1]).[F_2] \neq 0$ because $\varnothing \neq u_1(\Sigma_1^1) \subset u_2(\Sigma_2^1)$. Hence $E_1.F_2 = (u_1)_*([\Sigma_1]).[F_2] \neq 0$ which is a contradiction because E_1 and F_2 are in different fibers of π by property (3). Hence $(u_1)_*([\Sigma_1^i]).[F] = 0$ for all i.

Step 3: There is an effective divisor D_{F_1} whose support is F_1 and an integer $\kappa' > 0$ with the property that: $[D_{F_1}] + \kappa'(u_1)_*([\Sigma_1])$ represents $[F]$. Each irreducible component of the above curve has intersection number zero with $[F]$ by Step 2 hence by Corollary 6.6, we get that A is compactified k uniruled. $\qquad\square$

Lemma 6.9. *Let $A = X \setminus D$ be a smooth affine surface where X is a smooth projective variety and D is a connected smooth normal crossing divisor. Let $\pi : X \to B$ be a morphism of projective varieties so that the generic fiber is isomorphic to $\mathbb{P}^1$ and intersects D k times. Let E be a smooth divisor in X. Suppose that:*

(1) There is a nef divisor G whose support is in D such that $E.G = 0$.

(2) We have $E.E = -1$, $E.D = 1$ and E is not contained in D. This means that there is a unique irreducible curve D_E in D intersecting E. We will assume that $D_E.G \neq 0$.

(3) We have that $D_E \cup E$ is contained in a fiber $\pi^{-1}(b)$ and there is an effective divisor D_F whose support is $\pi^{-1}(b) \cap D$ and a natural number $\kappa > 0$ so that $[D_F] + \kappa[E]$ represents the homology class of a fiber of π.
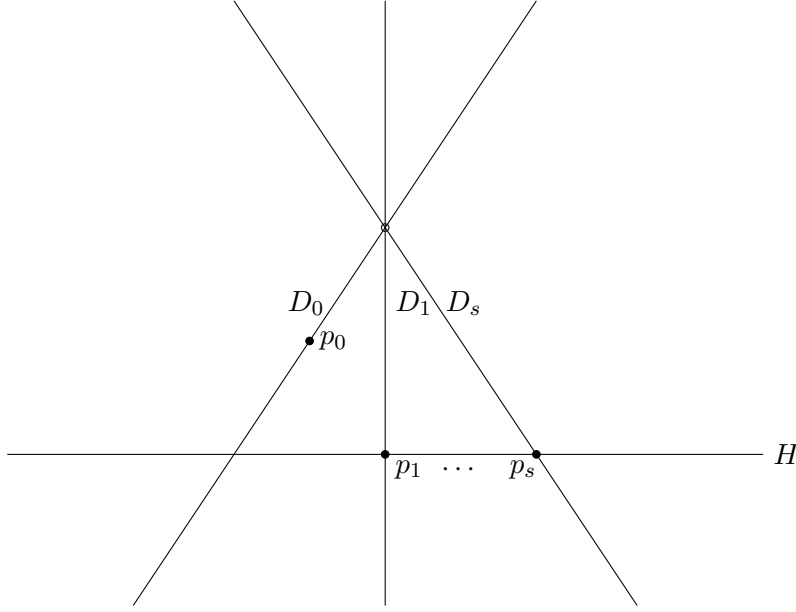
Then A is compactified k uniruled.

Proof. of Lemma 6.9. This proof will be done in two steps. In Step 1 we will show for any almost complex structure J compatible with the symplectic form on X and which is standard near D, the homology class $[E]$ can be represented by an irreducible J holomorphic curve. Finally in Step 2 we will use Corollary 6.6.

Step 1: By [McD90, Lemma 3.1] there is a J holomorphic map $u : \Sigma \to X$ from a connected genus 0 nodal Riemann surface Σ representing the exceptional class $[E_i] \in H_2(X)$. Let $\Sigma^1, \cdots, \Sigma^l$ be the irreducible components of Σ. In this step we want to show that $l = 1$. Because $u_*([\Sigma^i]).G \geq 0$ for all i, and that $u_*([\Sigma]).G = 0$ we then get $u_*([\Sigma^i]).G = 0$ for all i. Hence by property (2), $u(\Sigma^i)$ is not contained in D_E for any i. This means that $u_*([\Sigma^i]).D_E \geq 0$ for all i. The above statement combined with the fact that $D_E.E = 1$ means that there is exactly one irreducible component Σ^j intersecting D_E and this irreducible component intersects D_E with multiplicity 1. We may as well assume that $\Sigma^j = \Sigma^1$. Let D_X be an effective ample divisor whose support is D. Then $E.D_X = u_*([\Sigma]).D_X$, and $u_*([\Sigma^i]).D_X > 0$ for all i because A is an exact symplectic manifold. Let $c > 0$ be the coefficient of D_E in D_X. Then $cE.D_E = E.D_X$ because D_E is the only irreducible divisor in D intersecting E. Also because $u_*([\Sigma^1]).D_E = 1$ we get that $u_*([\Sigma^1]).D_X \geq c$. Hence $u_*([\Sigma^1]).(D_X - c.D_E) \geq 0$. Also for $i > 1$ we have that $u_*([\Sigma^i]).D_E = 0$ which implies that $u_*([\Sigma^i]).(D_X - cD_E) = u_*([\Sigma^i]).D_X > 0$. Hence if $l > 1$ we get that $u_*([\Sigma]).(D_X - cD_E) = \sum_i u_*([\Sigma^i]).(D_X - cD_E) > 0$ which contradicts the fact that $E.(D_X - cD_E) = 0$. This means that $l = 1$ and so Σ is irreducible.

Step 2: By Step 1 we have that Σ is irreducible and $u_*([\Sigma])$ represents E. Hence every irreducible component of the J holomorphic curve $D_F \cup u(\Sigma)$ has intersection number 0 with a fiber of B. So by property (3) combined with Corollary 6.6 we then get that A is compactified k uniruled.   □

We will give a fairly explicit description of acyclic surfaces of log Kodaira dimension 1. The constructions come from [GM88] (see also [Zai98, Theorem 2.6], [tDP90] and [FZ94]). We start with a line arrangement in $\mathbb{P}^2$ as in Figure 6.1.

Here we have curves $D_0, D_1, \cdots, D_s, H$ in this line arrangement. Let W be the divisor in $\mathbb{P}^2$ representing this line arrangement. At the point p_0 we blow up our surface many times according to the following rules:

(1) The first blow up must be at p_0.
(2) Each subsequent blow up must be on the exceptional divisor of the previous blow up and at a smooth point of the total transform of W.

At the point p_i where $i > 0$, we blow up in such a way as to to resolve the point of indeterminacy of $\frac{x^{m_i}}{y^{n_i}}$ (viewed as a birational map to $\mathbb{P}^1$) where x, y are local coordinates around p_i with $W = \{xy = 0\}$. These are chains of blowups where we only blow up along points p satisfying:

(1) p is in the exceptional divisor of the previous blowup.
(2) p is a nodal singular point of the total transform of W.

Let $E_0, E_1, \cdots, E_s$ be the last exceptional curves in these chains of blowups over $p_0, \cdots, p_s$. We let X be equal to $\mathbb{P}^2$ blown up as described above and we let D be the divisor in X equal to the total transform of W minus the last exceptional curves $E_0, \cdots, E_s$. Our surface A is equal to $X \setminus D$. The integers m_i, n_i are coprime and satisfy a certain equation to ensure that our surface A is affine and acyclic of log Kodaira dimension 1.

Lemma 6.10. *Suppose A is an acyclic surface of log Kodaira dimension 1. Then it is compactified 2 uniruled.*

Proof. of Lemma 6.10. We will use the notation $X, D_0, \cdots, D_s, E_0, \cdots, E_s$, $H, p_0, \cdots, p_s$, from before this lemma to describe A. We have three cases:

(1) $s = 1$.
(2) $s > 1$ and E_i intersects H for some i.
(3) $s > 1$ and E_i does not intersect H for any i.

Case (1): Let $\mathrm{Bl}_{p_0}(\mathbb{P}^2)$ be the blowup of $\mathbb{P}^2$ at the point p_0. We have a map from X to $\mathrm{Bl}_{p_0}(\mathbb{P}^2)$ which is a sequence of blowdown maps. We also have a fibration $\pi : \mathrm{Bl}_{p_0} \to \mathbb{P}^1$ whose fibers are proper transforms of lines in $\mathbb{P}^2$ passing through p_0. Let $\widetilde{\pi} : X \to \mathbb{P}^1$ be the composition of the map from X to $\mathrm{Bl}_{p_0}(\mathbb{P}^2)$ with π. Because $s = 1$, we have that a generic fiber of $\widetilde{\pi}$ intersects D twice.

Also, the proper transform of D_0 is a fiber of $\widetilde{\pi}$. So if we choose any almost complex structure J which is equal to the standard one near D, we have a fiber represented by the irreducible J holomorphic curve D_0. So, D_0 has intersection number 0 with any fiber. By Corollary 6.6 we then get that A is compactified 2 uniruled.

Case (2):

Let q be the point where all the divisors $D_0, \cdots, D_s$ intersect in one point and let $\widetilde{q}$ be the corresponding point in X. We blow up $\widetilde{q}$ giving us $\widetilde{X}$. Let $\widetilde{D}$ be the total transform of D, so $A = \widetilde{X} \setminus \widetilde{D}$. Note that $\widetilde{X}$ is equal to $\mathrm{Bl}_q \mathbb{P}^2$ blown up many times at the points $p_0, \cdots, p_s$. Hence we have a natural blowdown map $\mathrm{Bl}_{\widetilde{X}} : \widetilde{X} \to \mathrm{Bl}_q \mathbb{P}^2$. There is a natural map $\pi' : \mathrm{Bl}_q \mathbb{P}_2 \to \mathbb{P}^1$ whose fibers are proper transforms of lines passing through q. Let $\widetilde{\pi}' : \widetilde{X} \to \mathbb{P}^1$ be the composition $\pi' \circ \mathrm{Bl}_{\widetilde{X}}$. We let $\widetilde{E}_0, \cdots, \widetilde{E}_s$ be the proper transforms of $E_0, \cdots, E_s$ in $\widetilde{X}$ respectively. We similarly define $\widetilde{H}$, $\widetilde{D}_i$. Let E be the proper transform of the exceptional divisor of $\mathrm{Bl}_q \mathbb{P}^2$ in $\widetilde{X}$. The image of our morphism $\widetilde{\pi}'$ is $B := \mathbb{P}^1$. We define $b_j \in B$ so that $(\widetilde{\pi}')^{-1}(b_j)$ contains $\widetilde{E}_j$.

We have that $\widetilde{E}_i$ intersects $\widetilde{H}$ for some i. This is contained in some fiber $(\widetilde{\pi}')^{-1}(b_i)$. We have that $(\widetilde{\pi}')^{-1}(b_i)$ is obtained from $(\pi')^{-1}(b_i)$ by blowing up the point where this fiber intersects H repeatedly. Hence if R is the irreducible component of $(\widetilde{\pi}')^{-1}(b_i)$ that intersects E, then it is smooth and has self intersection -1. This means that $R + E$ is an effective nef divisor. We have that $\widetilde{E}_0.D = 1$ so let D_E be the unique divisor that intersects $\widetilde{E}_0$. Let D' be the union of irreducible curves in D not intersecting $\widetilde{E}_0$ and let Δ' be the connected component of D' containing $R \cup E$. Using Lemma 6.7 with the divisors $\Delta' + D_E$ and E there is a nef divisor G with the property that $G.D_E \neq 0$ and $G.E = 0$. The generic fiber of $\widetilde{\pi}'$ intersects $\widetilde{D}$ twice. Also $D_E \cup \widetilde{E}_0$ is contained in $(\widetilde{\pi}')^{-1}(b_0)$ and also there is an effective divisor D_F whose support is in $(\widetilde{\pi}')^{-1}(b_0) \cap D$ and $\kappa \in \mathbb{N}$ so that $[D_F] + \kappa[\widetilde{E}_0]$ is homologous to a fiber of $\widetilde{\pi}'$. So by Lemma 6.9 we get that A is compactified 2 uniruled.

Case (3):

Because $s > 1$ and E_i does not intersect H for all i, we get that E_1 and E_2 exist and do not intersect H. We have that $(\widetilde{\pi}')^{-1}(b_j)$ is a union of irreducible curves F_j in D plus $\widetilde{E}_j$. Also $\widetilde{E}_j.\widetilde{D} = \widetilde{E}_j.\widetilde{F}_j$ for all j and $\widetilde{D} \cup \widetilde{E}_1 \cup \widetilde{E}_2$ is a smooth normal crossing divisor. Let D'_j be equal to $\widetilde{D}$ minus the irreducible components of $\widetilde{D}$ intersecting E_j. We have that D'_j is

connected for each j and every irreducible component of $\widetilde{D}$ which intersects E_j also intersects D'_j.

We have that $\widetilde{D}$ is connected and $E \cup \widetilde{D}_0$ are disjoint from E_j for each j. Both E and $\widetilde{D}_0$ intersect each other and have self intersection -1 which implies that $G := E + \widetilde{D}_0$ is nef and contained in D'_j for each j. This does not intersect E_j. Hence by Lemma 6.8 we then get that A is compactified 2 uniruled. □

Proof. of Theorem 6.3. In order to prove this theorem we only need to show the following fact: *If A has log Kodaira dimension i where $i \leq 1$, then B has log Kodaira dimension $\leq i$.* This is because log Kodaira dimension is at most 2.

By [Fuj82] we have that the log Kodaira dimension of A is either $-\infty$, 1 or 2. Also if it is equal to $-\infty$ then $A = \mathbb{C}^2$. Suppose the log Kodaira dimension of A is $-\infty$ then $A = \mathbb{C}^2$. Also B is diffeomorphic to A and hence contractible and simply connected at infinity. By [Ram71] we then get that B is isomorphic to $\mathbb{C}^2$ and hence has log Kodaira dimension $-\infty$. Now suppose that A has log Kodaira dimension 1. By Lemma 6.10 we have that A is compactified 2 uniruled, so by Lemma 6.2, B has log Kodaira dimension ≤ 1. Putting everything together gives us that A and B must have the same log Kodaira dimension. □

6.2. Dimension 3.

Theorem 6.11. *Suppose that A is a smooth affine variety of dimension 3 such that A admits a compactification X with the following properties:*

(1) The compactification divisor D is smooth normal crossing and nef.
(2) The linear system $|D|$ contains a smooth member.

Let B be any smooth affine variety symplectomorphic to A and $\overline{\kappa}(A) = 2$ then $\overline{\kappa}(B) \leq 2$.

Proof. of Theorem 6.11. By [Kis06] we have that A admits a $\mathbb{C}^*$ fibration. In fact we can say more: [Kis06, Lemma 4.1,4.2,4.3] says that there is a projective variety X^s and a nef divisor D^s so that

(1) $A = X^s \setminus D^s$.
(2) There is a morphism $\pi : X^s \to W$ of projective varieties with the property that a generic fiber is isomorphic to $\mathbb{P}^1$ and intersects D^s twice.

By [Hir64] we can blow up X^s away from A giving us a smooth projective variety X and so that the total transform D of D^s is a smooth normal crossing divisor. Let $\widetilde{\pi} : X \to W$ be the composition of π with the blowdown map $X \to X^s$. Let D_X be the effective divisor which is the pullback of D^s under the blowdown map. We have that D_X is nef and that a generic fiber F of $\widetilde{\pi}$ satisfies $F.D_X = 2$. By Lemma 5.2, we then get that A is compactified 2 uniruled. Hence by Lemma 6.2, we get that the log Kodaira dimension of B is ≤ 2. □

7. Uniruledness of compactifications

If a projective variety X has a morphism $f : \mathbb{P}^1 \to X$ passing through every point $x \in X$ then we say that X is *uniruled*.

Theorem 7.1. *Suppose that two smooth projective varieties P and Q have affine open subsets A, B with the property that A is symplectomorphic to B. Then P is uniruled if and only if Q is.*

Proof. of Theorem 7.1. Suppose that P is uniruled, then we will show that Q is uniruled. Let D_P be an effective ample divisor whose support is $P \setminus A$ and D_Q an effective ample divisor whose support is $Q \setminus B$. By [Rua99] or [Kol98] we have that $\langle [\text{pt}], \alpha_1, \cdots, \alpha_k \rangle^P_{0,\beta} \neq 0$ for some $\beta \in H_2(P, \mathbb{Z})$ and cohomology classes $\alpha_1, \cdots, \alpha_k$. Let $k := \beta.D_P$. This means that any compatible J in P which is standard near D_P has the property that there is some J holomorphic curve $u : \Sigma \to P$ passing through any point p. Because D_P is nef, each irreducible component Σ_i of Σ satisfies $u_*(\Sigma_i).D_P \leq k$. In particular this is true for any irreducible component that passes through p. Hence A is compactified k-uniruled. So by Theorem 5.3, we have that the Liouville domain $\overline{A}$ is (k, Λ)-uniruled for some $\Lambda > 0$. Because the completion of $\overline{A}$ is symplectomorphic to the completion of $\overline{B}$ we then get by Theorem 2.5 that $\overline{B}$ is (k, Λ') uniruled for some $\Lambda' > 0$. Hence by Theorem 3.3, we have that B is algebraically k-uniruled. This implies that its compactification Q is uniruled. By symmetry, if Q is uniruled then P is. Hence P is uniruled if and only if Q is uniruled. $\qquad\square$

8. Appendix : plurisubharmonic functions on smooth affine varieties

The contents of this appendix are all contained inside the proof of [McL12, Lemma 2.1] and the ideas of that proof are contained in [Sei08, Section 4b]. We let A be a smooth affine variety. Here we recall the construction of the Liouville domain $\overline{A}$ (see Definition 3.1). Choose any algebraic embedding ι of A into $\mathbb{C}^N$ (so it is a closed subvariety). We have $\theta_A := -d^c R = \sum_i \frac{r_i^2}{2} d\vartheta_i$ where (r_i, ϑ_i) are polar coordinates for the ith $\mathbb{C}$ factor. We have that $d\theta_A$ is equal to the standard symplectic structure on $\mathbb{C}^N$. By abuse of notation we write θ_A for $\iota^* \theta_A$, and $\omega_A := d\theta_A$. Here $(\overline{A}, \theta_A) := (R^{-1}(-\infty, C], \theta_A)$ for $C \gg 0$.

We can also construct other Liouville domains as follows (see Definition 3.2): Let X be a smooth projective variety such that $X \setminus A$ is a smooth normal crossing divisor (an SNC compactification). Let L be an ample line bundle on X given by an effective divisor D whose support is $X \setminus A$. From now on such a line bundle will be called a **line bundle associated to an SNC compactification X of A**. Suppose $|\cdot|$ is some metric on L whose curvature form is a positive $(1,1)$ form. Then if s is some section of L such that $s^{-1}(0) = D$ then we define $\phi_{s,|\cdot|} := -\log |s|$ and $\theta_{s,|\cdot|} := -d^c \phi_{s,|\cdot|}$. The two form $d\theta_{s,|\cdot|}$ extends to a symplectic form $\omega_{|\cdot|}$ on X (which is independent

of s but does depend on $|\cdot|$). We will say that $\phi_{s,|\cdot|}$ is a *plurisubharmonic function associated to* L, $\theta_{s,|\cdot|}$ a **Liouville form associated** L and $\omega_{|\cdot|}$ a **symplectic form on** X **associated to** L. From [Sei08, Section 4b], we have that for $C \gg 1$,

$$(A_C, \theta_C) := (\phi_{s,|\cdot|}^{-1}(-\infty, C], \theta_{s,|\cdot|})$$

is a Liouville domain.

Let (r_i, ϑ_i) be polar coordinates for the i'th factor in $\mathbb{C}^N$.

Lemma 8.1. *If we compactify $\mathbb{C}^N$ by $\mathbb{P}^N$, there is a section S of $\mathcal{O}(1)$ and metric $\|\cdot\|$ with the following properties:*

(1) $-\log \|S\||_A$ is equal to $f(R)$ for some non-decreasing smooth function $f : \mathbb{R} \to \mathbb{R}$.

(2) $-\log \|S\||_A$ has no singularities near infinity.

Hence $R|_B$ has no singularities near infinity.

Proof. of Lemma 8.1. Let $H := \mathbb{P}^N \setminus \mathbb{C}^N$ and let S be a section of $\mathcal{O}(1)$ such that $S^{-1}(0) = H$. Let $\|\cdot\|$ be the standard Fubini Study metric on $\mathcal{O}(1)$. We have that $U(N+1)$ acts on $\mathbb{P}^N$ and it naturally lifts to an action on the total space of $\mathcal{O}(1)$. Let $U(N) \subset U(N+1)$ be the natural subgroup that preserves H. We have that $\|S\|$ is invariant under this action.

Because $-\log \|S\|$ is invariant under this action and exhausting, it is equal to $f(R)$ for some non decreasing smooth function $f : \mathbb{R} \to \mathbb{R}$. This is because $U(N)$ acts transitively on the level sets of R.

Let X be the closure of A in $\mathbb{P}^N$. By [Hir64] we can blow up $\mathbb{P}^N$ along H so that the proper transform $\widetilde{X}$ of X is smooth and the total transform $\widehat{H}$ of $H \cap X$ inside $\widetilde{X}$ is a smooth normal crossing divisor. We pull back the line bundle $\mathcal{O}(1)|_X$ to a line bundle L_X on $\widetilde{X}$ and also pull back the metric $\|\cdot\|$ and section S. We will write $\|\cdot\|_X$ and S_X for the new metric and section.

Let $p \in \widehat{H}$ and choose local holomorphic coordinates $z_1, \cdots, z_n$ on $\widetilde{X}$ and a trivialization of L_X around p so that $S_X = z_1^{w_1} \cdots z_n^{w_n}$ ($w_i \geq 0$). The metric $\|.\|_X$ on L_X is equal to $e^{\psi}|.|$ for some function ψ with respect to this trivialization where $|.|$ is the standard metric on $\mathbb{C}$. So

$$-d\log \|S_X\|_X = -d\psi - \left(\sum_i w_i d\log |z_i|\right).$$

If we take the vector field $Y := -r_1 \partial_{r_1} \cdots - r_n \partial_{r_n}$ (where $z_j = r_j e^{i\vartheta_j}$), then $d\log(|z_j|)(Y) = -1$ and $d\psi(Y)$ tends to zero. Hence $d\log \|S_X\|_X$ is non-zero near infinity which implies that $f(R)|_A = -\log \|S\|_X$ has no singularities near infinity. $\square$

Lemma 8.2. *For $C \gg 1$, (A_C, θ_C) is Liouville deformation equivalent to $(\overline{A}, \theta_A)$.*

Proof. of Lemma 8.2. By Lemma 8.1 we have that $R|_A$ has no singularities for $R \geq C$. Let $c \geq C$ and write $A'_c := (R|_A)^{-1}(-\infty, c]$. Because c is a

regular value of $R|_A$, we have that A'_c is a Liouville domain and by definition it is equal to $(\overline{A}, \theta_A)$.

Let S and $\|\cdot\|$ be the section and metric on $\mathcal{O}(1)$ coming from Lemma 8.1. We also have that $f(R) = -\log\|S\|$ where f is a smooth function with positive derivative when R is large. Let $A''_c := (-\log\|S\|)^{-1}(-\infty, c]$. We have that $A'_{f(c)} = A''_c$ for $c \gg 1$. We also have that $t\theta_A + (1-t)\theta_{S,\|\cdot\|}|_A$ is a deformation of Liouville domains from $(A'_{f(c)}, \theta_A)$ to $(A''_c, \theta_{S,\|\cdot\|})$.

Let $\phi_{s,|\cdot|}$ be a plurisubharmonic function associated to our line bundle L so that $A_C = \phi_{s,|\cdot|}^{-1}(-\infty, C]$. Let $\phi_t := (1-t)\phi_{s,|\cdot|} - t\log\|S\|$ and let $A^t_c := \phi_t^{-1}(-\infty, c]$. By using work from [Sei08, Section 4b], we have for C large enough that $(A^t_C, -d^c\phi_t)$ is a Liouville deformation from $(A''_C, \theta_{S,\|\cdot\|})$ to (A_C, θ_C). Hence by composing the above two Liouville deformations, we get that (A_C, θ_C) is Liouville deformation equivalent to $(\overline{A}, \theta_A)$ for C large enough. $\qquad\square$

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